Unit 1 Integration: Standard Integration

- The process of integration as a reverse of differentiation
- Is also a process of summation or adding parts
- The process is denoted by the variable of integration

  the integral sign the function to be integrated

Hence

- The general solution of integrals of the form where a and n are constants real numbers(fractions, zero, a positive and negative) with the exception of n.
- where c is called a constant of integration

Standard Integration

- Integrating a constant
- Integrating a sum of expressions= to the integration of the individual expressions.

- Because there are no limits placed on the variable of integration the result has a constant c the integral is called **an indefinite integral.**
- **Summary standard integrals from differentiation**

(i) $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$ (except when $n = -1$)

(ii) $\int \cos ax\,dx = \frac{1}{a}\sin ax + c$

(iii) $\int \sin ax\,dx = -\frac{1}{a}\cos ax + c$

(iv) $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + c$

(v) $\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + c$

(vi) $\int \operatorname{cosec} ax \cot ax\,dx = -\frac{1}{a}\operatorname{cosec} ax + c$

(vii) $\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + c$

(viii) $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$

(ix) $\int \frac{1}{x}\,dx = \ln x + c$

Examples

1. Determine solution:

Rearranging into standard integral form gives:

$$\int \frac{2x^3 - 3x}{4x}\,dx = \int \frac{2x^3}{4x} - \frac{3x}{4x}\,dx$$

$$= \int \frac{x^2}{2} - \frac{3}{4}\,dx = \left(\frac{1}{2}\right)\frac{x^{2+1}}{2+1} - \frac{3}{4}x + c$$

$$= \left(\frac{1}{2}\right)\frac{x^3}{3} - \frac{3}{4}x + c = \frac{1}{6}x^3 - \frac{3}{4}x + c$$

2. Determine Solution:

$$\int \frac{-5}{9\sqrt[4]{t^3}}\,dt = \int \frac{-5}{9t^{\frac{3}{4}}}\,dt = \int \left(-\frac{5}{9}\right)t^{-\frac{3}{4}}\,dt \quad = \left(-\frac{5}{9}\right)\frac{t^{-\frac{3}{4}+1}}{-\frac{3}{4}+1} + c$$

3. Determine solution:

$$\int \frac{(1+\theta)^2}{\sqrt{\theta}}\,d\theta = \int \frac{(1+2\theta+\theta^2)}{\sqrt{\theta}}\,d\theta \quad = \int \left(\frac{1}{\theta^{\frac{1}{2}}} + \frac{2\theta}{\theta^{\frac{1}{2}}} + \frac{\theta^2}{\theta^{\frac{1}{2}}}\right)d\theta$$

$$= \int \left(\theta^{\frac{-1}{2}} + 2\theta^{1-\left(\frac{1}{2}\right)} + \theta^{2-\left(\frac{1}{2}\right)}\right)d\theta \quad = \int \left(\theta^{\frac{-1}{2}} + 2\theta^{\frac{1}{2}} + \theta^{\frac{3}{2}}\right)d\theta$$

$$= \frac{\theta^{\left(\frac{-1}{2}\right)+1}}{-\frac{1}{2}+1} + \frac{2\theta^{\left(\frac{1}{2}\right)+1}}{\frac{1}{2}+1} + \frac{\theta^{\left(\frac{3}{2}\right)+1}}{\frac{3}{2}+1} + c \quad = \frac{\theta^{\frac{1}{2}}}{\frac{1}{2}} + \frac{2\theta^{\frac{3}{2}}}{\frac{3}{2}} + \frac{\theta^{\frac{5}{2}}}{\frac{5}{2}} + c$$

$$= 2\theta^{\frac{1}{2}} + \frac{4}{3}\theta^{\frac{3}{2}} + \frac{2}{5}\theta^{\frac{5}{2}} + c$$

Determin (a) $\int \left(\frac{2m^2+1}{m}\right) dm$ (b) $\int 3\cos 2x\, dx$ (c) $\int 7\sin 3\theta\, d\theta$

Definite Integral

- When the integral sign's top and bottom has values they represent what are called the limits of the variable. The integral is now called **definite integral**.

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- After integration a substitution of variable in the resulting function with the limits

a is called the lower limit , b is called the upper limit

- Examples : (a) $\int_1^2 4e^{2x}\, dx$ (b) $\int_{-1}^{2} \frac{2}{3e^{2x}}\, dx$ (c) $\int_0^1 5\cos 3x\, dx$ (d) $\int_0^{\frac{\pi}{6}} 3\sec^2 2x\, dx$

Algebraic Substitution

- The substitution made introduces a new variable e g. u or t .
- The resulting integral f(u)du
- It applies to integrals of the form
- Example 1. Determine : Example 2

Solution

Evaluate

Solution:

Let $u = (2x \quad 5)$ then $\frac{du}{dx} = 2$ and $dx = \frac{du}{2}$

Hence

$$\int (2x-5)^7\,dx = \int u^7 \frac{du}{2} = \frac{1}{2}\int u^7\,du$$

$$= \frac{1}{2}\left(\frac{u^8}{8}\right) + c = \frac{1}{16}u^8 + c$$

Rewriting u as $(2x-5)$ gives:

$$\int (2x-5)^7\,dx = \frac{1}{16}(2x-5)^8 + c$$

Let $u = 6x - 1$ then $\frac{du}{dx} = 6$ and $dx = \frac{du}{6}$

Hence $$\int 2e^{6x-1}\,dx = \int 2e^u \frac{du}{6} = \frac{1}{3}\int e^u\,du$$

$$= \frac{1}{3}e^u + c = \frac{1}{3}e^{6x-1} + c$$

Thus $$\int_0^1 2e^{6x-1}\,dx = \frac{1}{3}[e^{6x-1}]_0^1$$

$$= \frac{1}{3}[e^5 - e^{-1}] = 49.35,$$

correct to 4 significant figures.

Examples

3. Determine Solution:

Let $u = (4x^2 + 3)$ then $\frac{du}{dx} = 8x$ and $dx = \frac{du}{8x}$

Hence

$$\int 3x(4x^2 + 3)^5\,dx = \int 3x(u)^5 \frac{du}{8x}$$

$$= \frac{3}{8}\int u^5\,du, \text{ by cancelling}$$

Hence $\frac{3}{8}\int u^5\,du = \frac{3}{8}\left(\frac{u^6}{6}\right) + c = \frac{1}{16}u^6 + c \quad = \frac{1}{16}(4x^2 + 3)^6 + c$

4. Evaluate

Solution

Let $u = \sin\theta$ then $\frac{du}{d\theta} = \cos\theta$ and $d\theta = \frac{du}{\cos\theta}$

Hence $\int 24\sin^5\theta\cos\theta\,d\theta \quad = \int 24u^5\cos\theta\frac{du}{\cos\theta}$

$$= 24\int u^5\,du, \text{ by cancelling}$$

$$= 24\frac{u^6}{6} + c = 4u^6 + c = 4(\sin\theta)^6 + c$$

$$= 4\sin^6\theta + c$$

Thus $\int_0^{\pi/6} 24\sin^5\theta\cos\theta\,d\theta \quad = [4\sin^6\theta]_0^{\pi/6} = 4\left[\left(\sin\frac{\pi}{6}\right)^6 - (\sin 0)^6\right]$

$$= 4\left[\left(\frac{1}{2}\right)^6 - 0\right] = \frac{1}{16} \text{ or } 0.0625$$

Questions

1. Find 2. Determine 3. Show that

4. Integrate w.r.t the variable

5.Evaluate the definite integrals

(a)

- **NB When evaluating definite integral by substitution it is faster to also change the limits to the new variable.**

Trigonometrical and the t=tan

- Summary trigonometric integrals and substitutions

	$f(x)$	$\int f(x)dx$	Method
1.	$\cos^2 x$	$\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)+c$	Use $\cos 2x = 2\cos^2 x - 1$
2.	$\sin^2 x$	$\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)+c$	Use $\cos 2x = 1 - 2\sin^2 x$
3.	$\tan^2 x$	$\tan x - x + c$	Use $1 + \tan^2 x = \sec^2 x$
4.	$\cot^2 x$	$-\cot x - x + c$	Use $\cot^2 x + 1 = \operatorname{cosec}^2 x$
5.	$\cos^m x \sin^n x$	(a) If either m or n is odd (but not both), use $\cos^2 x + \sin^2 x = 1$ (b) If both m and n are even, use either $\cos 2x = 2\cos^2 x - 1$ or $\cos 2x = 1 - 2\sin^2 x$	
6.	$\sin A \cos B$		Use $\frac{1}{2}[\sin(A+B) + \sin(A-B)]$
7.	$\cos A \sin B$		Use $\frac{1}{2}[\sin(A+B) - \sin(A-B)]$
8.	$\cos A \cos B$		Use $\frac{1}{2}[\cos(A+B) + \cos(A-B)]$
9.	$\sin A \sin B$		Use $-\frac{1}{2}[\cos(A+B) - \cos(A-B)]$

- Substitutions cont’d

10.	$\dfrac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\dfrac{x}{a} + c$	Use $x = a \sin\theta$ substitution
11.	$\sqrt{a^2 - x^2}$	$\dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a} + \dfrac{x}{2}\sqrt{a^2 - x^2} + c$	

Examples

1. Determine

Soln:

Since $\cos 2x \equiv 1 - 2\sin^2 x$ then $\sin^2 x \equiv \frac{1}{2}(1 - \cos 2x)$ and

$$\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$$

$$\text{Hence} \quad \int \sin^2 3x\, dx = \int \frac{1}{2}(1 - \cos 6x)\, dx$$

$$= \frac{1}{2}\left(x - \frac{\sin 6x}{6}\right) + c$$

2. Evaluate

Since $\cos 2t = 2\cos^2 t - 1$

then $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$ and

$$\cos^2 4t = \frac{1}{2}(1 + \cos 8t)$$

Hence $\int_0^{\frac{\pi}{4}} 2\cos^2 4t\, dt$

$$= 2\int_0^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos 8t)\, dt = \left[t + \frac{\sin 8t}{8}\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{4} + \frac{\sin 8\left(\frac{\pi}{4}\right)}{8}\right] - \left[0 + \frac{\sin 0}{8}\right]$$

$$= \frac{\pi}{4} \text{ or } 0.7854$$

3. Evaluate

Soln:

Since $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$, then
$\cot^2\theta = \operatorname{cosec}^2\theta - 1$ and $\cot^2 2\theta = \operatorname{cosec}^2 2\theta - 1$

$$\text{Hence} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}\cot^2 2\theta\, d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\operatorname{cosec}^2 2\theta - 1)\, d\theta = \frac{1}{2}\left[\frac{-\cot 2\theta}{2} - \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[\left(\frac{-\cot 2\left(\frac{\pi}{3}\right)}{2} - \frac{\pi}{3}\right) - \left(\frac{-\cot 2\left(\frac{\pi}{6}\right)}{2} - \frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}[(- - 0.2887 - 1.0472) - (-0.2887 - 0.5236)]$$

$$= 0.0269$$

4.Determine

Soln:

Since $\cos^2\theta + \sin^2\theta = 1$ then $\sin^2\theta = (1 - \cos^2\theta)$.

Hence $\int \sin^5\theta\, d\theta$

$$= \int \sin\theta(\sin^2\theta)^2\, d\theta = \int \sin\theta(1-\cos^2\theta)^2\, d\theta$$

$$= \int \sin\theta(1 - 2\cos^2\theta + \cos^4\theta)\, d\theta$$

$$= \int (\sin\theta - 2\sin\theta\cos^2\theta + \sin\theta\cos^4\theta)\, d\theta$$

$$= -\cos\theta + \frac{2\cos^3\theta}{3} - \frac{\cos^5\theta}{5} + c$$

5. Evaluate

Soln:

$$\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x\, dx = \int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x \cos x\, dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 x)(1 - \sin^2 x)(\cos x)\, dx = \int_0^{\frac{\pi}{2}} (\sin^2 x \cos x - \sin^4 x \cos x)\, dx$$

$$= \left[\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}\right]_0^{\frac{\pi}{2}} = \left[\frac{\left(\sin\frac{\pi}{2}\right)^3}{3} - \frac{\left(\sin\frac{\pi}{2}\right)^5}{5}\right] - [0 - 0]$$

$$= \frac{1}{3} - \frac{1}{5} = \frac{2}{15} \text{ or } 0.1333$$

NB Take note in example 4 that if a power of sin multiplies cos of power 1 or vice versa the integral can be obtained by inspection

Examples 6. Evaluate 7. Find

$$\int_0^{\frac{\pi}{4}} 4\cos^4\theta\,d\theta = 4\int_0^{\frac{\pi}{4}} (\cos^2\theta)^2\,d\theta = 4\int_0^{\frac{\pi}{4}} \left[\frac{1}{2}(1+\cos 2\theta)\right]^2 d\theta$$

$$= \int_0^{\frac{\pi}{4}} (1+2\cos 2\theta+\cos^2 2\theta)\,d\theta = \int_0^{\frac{\pi}{4}} \left[1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{3}{2}+2\cos 2\theta+\frac{1}{2}\cos 4\theta\right) d\theta = \left[\frac{3\theta}{2}+\sin 2\theta+\frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{3}{2}\left(\frac{\pi}{4}\right)+\sin\frac{2\pi}{4}+\frac{\sin 4(\pi/4)}{8}\right]-[0]$$

$$= \frac{3\pi}{8}+1$$

$= 2.178$, correct to 4 significant figures.

$$\int \sin 3t\cos 2t\,dt = \int \frac{1}{2}[\sin(3t+2t)+\sin(3t-2t)]\,dt,$$

$$= \frac{1}{2}\int (\sin 5t+\sin t)\,dt$$

$$= \frac{1}{2}\left(\frac{-\cos 5t}{5}-\cos t\right)+c$$

$$\int \frac{1}{3}\cos 5x\sin 2x\,dx$$

$$= \frac{1}{3}\int \frac{1}{2}[\sin(5x+2x)-\sin(5x-2x)]\,dx,$$

$$= \frac{1}{6}\int (\sin 7x-\sin 3x)\,dx$$

$$= \frac{1}{6}\left(\frac{-\cos 7x}{7}+\frac{\cos 3x}{3}\right)+c$$

Example 9. Determine

Let $x = a\sin\theta$ then $\dfrac{dx}{d\theta} = a\cos\theta$ and $dx = a\cos\theta\, d\theta$

Hence $\int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - a^2\sin^2\theta}\,(a\cos\theta\, d\theta)$

$= \int \sqrt{a^2(1 - \sin^2\theta)}\,(a\cos\theta\, d\theta) = \int \sqrt{a^2\cos^2\theta}\,(a\cos\theta\, d\theta)$

$= \int (a\cos\theta)(a\cos\theta\, d\theta) = a^2\int \cos^2\theta\, d\theta = a^2\int \left(\dfrac{1 + \cos 2\theta}{2}\right) d$

(since $\cos 2\theta \equiv 2\cos^2\theta - 1$)

$= \dfrac{a^2}{2}\left(\theta + \dfrac{\sin 2\theta}{2}\right) + c = \dfrac{a^2}{2}\left(\theta + \dfrac{2\sin\theta\cos\theta}{2}\right) + c$

since $\sin 2\theta = 2\sin\theta\cos\theta$

$= \dfrac{a^2}{2}[\theta + \sin\theta\cos\theta] + c$

Since $x = a\sin\theta$, then $\sin\theta = \dfrac{x}{a}$ and $\theta = \sin^{-1}\dfrac{x}{a}$

Also, $\cos^2\theta + \sin^2\theta = 1$, from which,

$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \left(\dfrac{x}{a}\right)^2} = \sqrt{\dfrac{a^2 - x^2}{a^2}} = \dfrac{\sqrt{a^2 - x^2}}{a}$

Thus $\int \sqrt{a^2 - x^2}\, dx = \dfrac{a^2}{2}[\theta + \sin\theta\cos\theta]$

$= \dfrac{a^2}{2}\left[\sin^{-1}\dfrac{x}{a} + \left(\dfrac{x}{a}\right)\dfrac{\sqrt{a^2 - x^2}}{a}\right] + c$

$= \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a} + \dfrac{x}{2}\sqrt{a^2 - x^2} + c$

Examples 10 Determine 11. Evaluate

Solution

Let $x = a\tan\theta$ then $\frac{dx}{d\theta} = a\sec^2\theta$ and $dx = a\sec^2\theta d\theta$

Hence $\int \frac{1}{(a^2+x^2)}dx = \int \frac{1}{(a^2+a^2\tan^2\theta)}(a\sec^2\theta\, d\theta)$

$= \int \frac{a\sec^2\theta\, d\theta}{a^2(1+\tan^2\theta)} = \int \frac{a\sec^2\theta\, d\theta}{a^2\sec^2\theta}$ since $1+\tan^2\theta \equiv \sec^2\theta$

$= \int \frac{1}{a}d\theta = \frac{1}{a}(\theta) + c$ Since $x = a\tan\theta$, $\theta = \tan^{-1}\frac{x}{a}$

Hence $\int \frac{1}{(a^2+x^2)}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

$$\int_0^1 \frac{5}{(3+2x^2)}dx = \int_0^1 \frac{5}{2[(3/2)+x^2]}dx$$

$$= \frac{5}{2}\int_0^1 \frac{1}{[\sqrt{3/2}]^2+x^2}dx - \frac{5}{2}\left[\frac{1}{\sqrt{3/2}}\tan^{-1}\frac{x}{\sqrt{3/2}}\right]_0^1$$

$$= \frac{5}{2}\sqrt{\frac{2}{3}}\left[\tan^{-1}\sqrt{\frac{2}{3}} - \tan^{-1}0\right] = (2.0412)[0.6847-0]$$

$= 1.3976$, correct to 4 decimal places

Questions 1. Determine

3.Determine

4.Evaluate the following definite integrals (a)

(b)

(e) [24]

5. Determine (a) (b)

Using the substitution t=

- Is used to solve integrals of the form
- and cos
- Since cos
- Sin
- Since there is a change of variable

1. Determine

If $t = \tan\frac{\theta}{2}$ then $\sin\theta = \frac{2t}{1+t^2}$ and $d\theta = \frac{2\,dt}{1+t^2}$

Thus $\int \frac{d\theta}{\sin\theta} = \int \frac{1}{\sin\theta}\,d\theta$

$= \int \frac{1}{\frac{2t}{1+t^2}}\left(\frac{2\,dt}{1+t^2}\right) = \int \frac{1}{t}\,dt = \ln t + c$

Hence $\int \frac{d\theta}{\sin\theta} = \ln\left(\tan\frac{\theta}{2}\right) + c$

2.

If $\tan\frac{x}{2}$ then $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2\,dt}{1+t^2}$ from equations (2) and (3).

Thus $\int \frac{dx}{1+\cos x} = \int \frac{1}{1+\cos x}\,dx = \int \frac{1}{1+\frac{1-t^2}{1+t^2}}\left(\frac{2\,dt}{1+t^2}\right)$

$= \int \frac{1}{(1+t^2)+(1-t^2)}\left(\frac{2\,dt}{1+t^2}\right) = \int dt$

Hence $\int \frac{dx}{1+\cos x} = t + c = \tan\frac{x}{2} + c$

3.

If $t = \tan\frac{\theta}{2}$ then $\cos\theta = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2\,dt}{1+t^2}$

Thus $\int \frac{d\theta}{5+4\cos\theta} = \int \frac{\left(\frac{2\,dt}{1+t^2}\right)}{5+4\left(\frac{1-t^2}{1+t^2}\right)}$

$= \int \frac{\left(\frac{2\,dt}{1+t^2}\right)}{\frac{5(1+t^2)+4(1-t^2)}{1+t^2}} = 2\int \frac{dt}{t^2+9} = 2\int \frac{dt}{t^2+3^2}$

$= 2\left(\frac{1}{3}\tan^{-1}\frac{t}{3}\right) + c.$

$\int \frac{d\theta}{5+4\cos\theta} = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{\theta}{2}\right) + c$

4. Determine

If $\tan\frac{x}{2}$ then $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2\,dt}{1+t^2}$ from equations (2) and (3).

Thus $$\int \frac{dx}{\cos x} = \int \frac{1}{\frac{1-t^2}{1+t^2}} \left(\frac{2\,dt}{1+t^2}\right)$$

$$= \int \frac{2}{1-t^2}\,dt$$

$\frac{2}{1-t^2}$ may be resolved into partial fractions

Let $$\frac{2}{1-t^2} = \frac{2}{(1-t)(1+t)}$$

$$= \frac{A}{(1-t)} + \frac{B}{(1+t)} = \frac{A(1+t)+B(1-t)}{(1-t)(1+t)}$$

Hence $$2 \equiv A(1+t) + B(1-t)$$

When $t = 1$, $2 = 2A$, from which, $A = 1$

When $t = -1$, $2 = 2B$, from which, $B = 1$

Hence $$\int \frac{2\,dt}{1-t^2} = \int \frac{1}{(1-t)} + \frac{1}{(1+t)}\,dt$$

$$= -\ln(1-t) + \ln(1+t) + c$$

$$= \ln\left\{\frac{(1+t)}{(1-t)}\right\} + c$$

Thus $$\int \frac{dx}{\cos x} = \ln\left\{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right\} + c$$

Note that since $\tan\frac{\pi}{4} = 1$, the above result may be written as:

$$\int \frac{dx}{\cos x} = \ln\left\{\frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{x}{2}}\right\} + c$$

$$= \ln\left\{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\} + c$$

from compound angles.

Use of Partial Fractions

- Summary of concept of partial frac

Type	Denominator containing	Expression	Form of partial fraction
1	Linear factors	$\frac{f(x)}{(x+a)(x-b)(x+c)}$	$\frac{A}{(x+a)}+\frac{B}{(x-b)}+\frac{C}{(x+c)}$
2	Repeated linear factors	$\frac{f(x)}{(x+a)^3}$	$\frac{A}{(x+a)}+\frac{B}{(x+a)^2}+\frac{C}{(x+a)^3}$
3	Quadratic factors	$\frac{f(x)}{(ax^2+bx+c)(x+d)}$	$\frac{Ax+B}{(ax^2+bx+c)}+\frac{C}{(x+d)}$

- Then find A ,B and C
- Example 1. Determine

Let $\frac{1}{(x^2-a^2)} \equiv \frac{A}{(x-a)}+\frac{B}{(x+a)}$

$\equiv \frac{A(x+a)+B(x-a)}{(x+a)(x-a)}$

Equating the numerators gives:

$1 \equiv A(x+a)+B(x-a)$

Let $x=a$, then $A=\frac{1}{2a}$, and let $x=-a$, then $B=-\frac{1}{2a}$

Hence $\int \frac{1}{(x^2-a^2)}\,dx \equiv \int \frac{1}{2a}\left[\frac{1}{(x-a)}-\frac{1}{(x+a)}\right]dx$

$= \frac{1}{2a}[\ln(x-a)-\ln(x+a)]+c$

$= \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)+c$

Example 2. Determine
Express in partial fractions

The denominator is a combination of a linear factor and a repeated linear factor.

Let $\dfrac{5x^2 - 2x - 19}{(x+3)(x-1)^2}$

$$\equiv \frac{A}{(x+3)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$\equiv \frac{A(x-1)^2 + B(x+3)(x-1) + C(x+3)}{(x+3)(x-1)^2}$$

by algebraic addition

Equating the numerators gives:

$$5x^2 - 2x - 19 \equiv A(x-1)^2 + B(x+3)(x-1) + C(x+3) \qquad (1)$$

Let $x = -3$. Then $5(-3)^2 - 2(-3) - 19 \equiv A(-4)^2 + B(0)(-4) + C(0)$

i.e. $32 \equiv 16A$

i.e. $A = 2$

Let $x = 1$. Then

$5(1)^2 - 2(1) - 19 \equiv A(0)^2 + B(4)(0) + C(4)$

i.e. $-16 \equiv 4C$ i.e. $C = -4$

Without expanding the RHS of equation (1) it can be seen that equating the coefficients of x^2 gives: $5 = A + B$, and since $A = 2$, $B = 3$

[Check: Identity (1) may be expressed as:

$$5x^2 - 2x - 19 \equiv A(x^2 - 2x + 1) + B(x^2 + 2x - 3) + C(x + 3)$$

i.e. $5x^2 - 2x - 19 \equiv Ax^2 - 2Ax + A + Bx^2 + 2Bx - 3B + Cx + 3C$

Equating the x term coefficients gives: $-2 \equiv -2A + 2B + C$

When $A = 2$, $B = 3$ and $C = -4$ then

$-2A + 2B + C = -2(2) + 2(3) - 4 = -2 \equiv$ LHS

Equating the constant term gives:

$$-19 \equiv A - 3B + 3C$$

$$\text{RHS} = 2 - 3(3) + 3(-4) = 2 - 9 - 12$$

$$\equiv -19 \equiv \text{LHS}]$$

Hence $\dfrac{5x^2 - 2x - 19}{(x+3)(x-1)^2}$

$$\equiv \frac{2}{(x+3)} + \frac{3}{(x-1)} - \frac{4}{(x-1)^2}$$

Questions 1. Determine

3.

Integration by parts

- Is derived by reversing the product rule of differentiation
- The product rule of differentiation
- Integrate w.r.t x

Gives i.e

Example 1 Determine

Repeat

Let $u = x^2$, from which, $\frac{du}{dx} = 2x$, i.e. $du = 2x\,dx$,
and let $dv = \sin x\,dx$, from which,

$$v = \int \sin x\,dx = -\cos x$$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int x^2 \sin x\,dx = (x^2)(-\cos x) - \int (-\cos x)(2x\,dx)$$

$$= -x^2 \cos x + 2\left[\int x \cos x\,dx\right]$$

$$= -x^2 \cos x + 2\{x \sin x + \cos x\} + c$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$$

$$= (2 - x^2)\cos x + 2x \sin x + c$$

Question 1 Find :

NB : In most situations integration by parts is done in such a way that the selection of u and dv/dx reduces to a standard . Here it is immaterial which part is selected as u or dv/dx.

Question 2 Evaluate

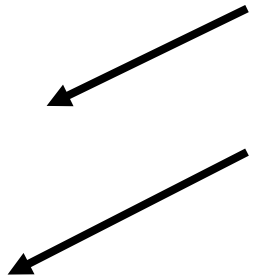
Question 3 Evaluate

Integration by Parts-Bernoulli's Generalized formula

- This formula is an extension of the formula of integration by parts.

  i.e

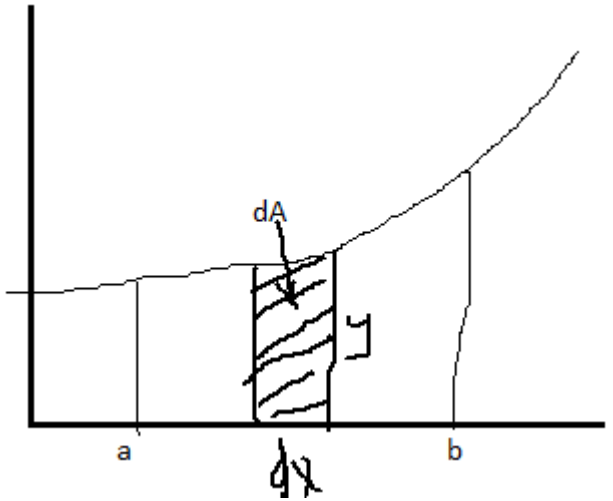
- Then selecting u and
- etc
- Applying rule for each stage



The process is repeated until

The method is applied **only one** of the function is a polynomial and make it u

Application 1: Area under Curve



- The small area A – (1)
- The accuracy will increase as the width

Smaller.

- The total area A =sum of the strips=
- From (1) -(2)
- Thus integrating (2)
- If area is required between specified limits a and b
  - Area A= {between curve and x axis}
  - Area A = {between curve and y axis
- NB Area in the negative of axis is also negative

Question 1 Find area enclosed by the curve y=sin2x, the x-axis and the ordinates x=-π/3 and π/3

Question 2 Show by integrating that the area of the triangle formed by the line y=2x, the ordinates x=0 and x=4 and the x-axis is 16 square units.

Question 3 A gas expands according to the law pv=constant. When volume is 3 when volume changes from 2

Question 4 Determine area enclosed between the curves y=+1 and y =7x

Question 5 Determine the area enclosed by the two curves y= and

Question 6 Determine the area enclosed by the straight lines y=4-x, y=3x and 3y=x

NB :In answering these questions a sketch of the curves is helpful

Application 2: Volume of Solids of Revolution

- The volume of solid of revolution is the volume of solid that is generated when the area is rotated about an axis for 36
- For a small element thickness ,

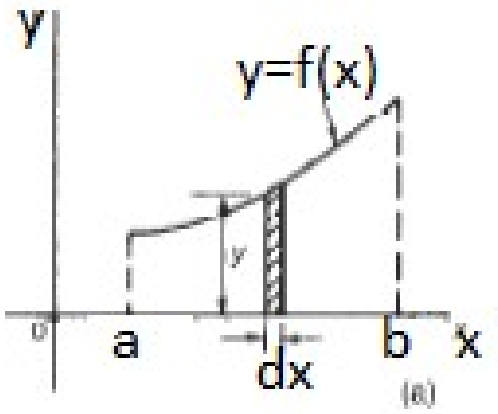
1 rev of rotation generates a small disc of radius y, thickness

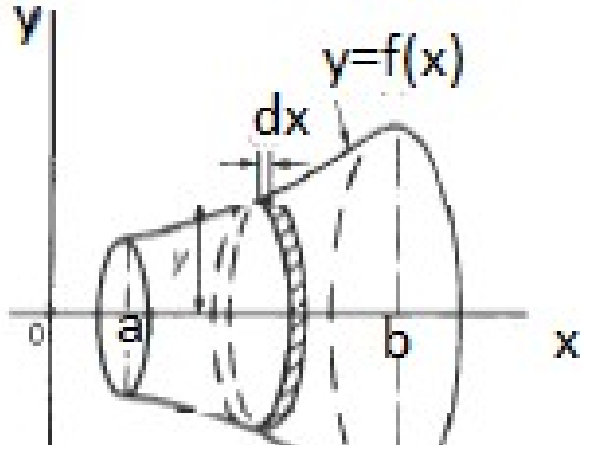
- Between the limits a and b

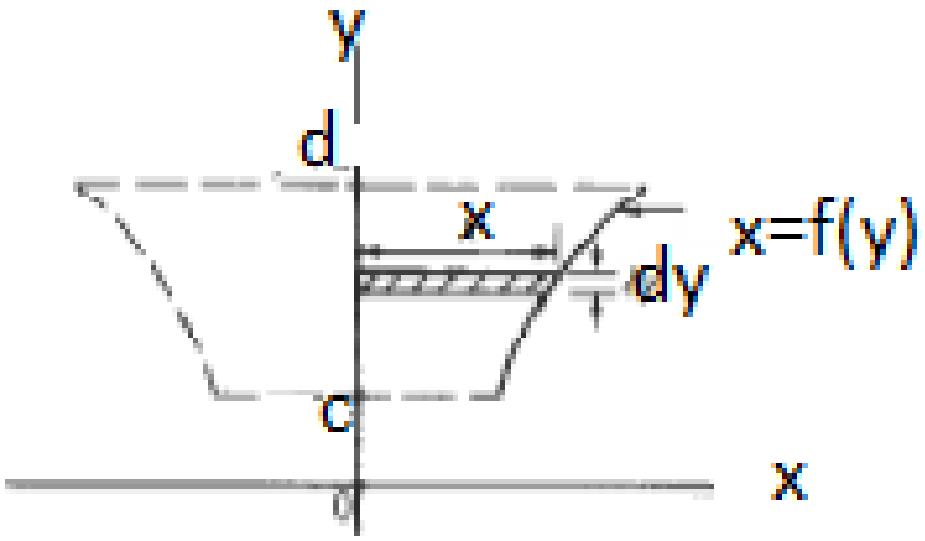
Volume =

- For rotation about the y-axis

Volume =







Examples 1 Find the volume of the solid of revolution when the curve y=2x is rotated one revolution about the x-axis between the limits

$$\text{volume generated} = \int_0^5 \pi y^2\,dx = \int_0^5 \pi(2x)^2\,dx$$

$$= \int_0^5 4\pi x^2\,dx = 4\pi\left[\frac{x^3}{3}\right]_0^5 = \frac{500\pi}{3} = \mathbf{166\frac{2}{3}\pi \text{ cubic units}}$$

2.The curve v= between x=1 and x=4. Determine the volume of t

$$\text{Volume} = \int_1^4 \pi y^2\,dx = \int_1^4 \pi(x^2+4)^2\,dx$$

$$= \int_1^4 \pi(x^4+8x^2+16)\,dx = \pi\left[\frac{x^5}{5}+\frac{8x^3}{3}+16x\right]_1^4$$

$$= \pi[(204.8+170.67+64) - (0.2+2.67+16)]$$

$$= \mathbf{420.6\pi \text{ cubic units}}$$

3. The area enclosed by the curve y=, is rotated one rev about

d 3. D me

$$\text{volume generated} = \int_{-1}^{3} \pi y^2\,dx = \int_{-1}^{3} \pi\left(3e^{\frac{x}{3}}\right)^2 dx$$

$$= 9\pi \int_{-1}^{3} e^{\frac{2x}{3}}\,dx$$

$$= 9\pi\left[\frac{e^{\frac{2x}{3}}}{\frac{2}{3}}\right]_{-1}^{3} = \frac{27\pi}{2}\left(e^2 - e^{-\frac{2}{3}}\right)$$

$$= \mathbf{92.82\pi \text{ cubic units}}$$

Questions 1 Determine the volume of a plug formed by a frustum of a sphere of radius 6cm which lies between two parallel planes at 2 cm and 4cm from the centre and on the same side of it Answer 53.33π

2. The area enclosed between the two curves is rotated about the x axis Determine the volume of solid of revolution.(8π)

3. The portion of the curve y= lying between x =1 and 3 is rotated one rev about x-axis. Determine the volume of revolution. (57.07π)

4. The area enclosed by the curves y=is rotated one rev about x-axis and y axis. Calculate the volume of solid of revolution (1326π) for x

Application 3: Mean and Root Mean Square Values

(a) The Mean or Average value is the height y such that

Example 1 A sinusoidal voltage is given by v=100sin volts, Determine the mean value of voltage over a half cycle using integration.

Half a cycle means the limits are 0 to π radians.
Mean value,

$$\bar{v} = \frac{1}{\pi - 0}\int_0^{\pi} v\, d(\omega t) = \frac{1}{\pi}\int_0^{\pi} 100\sin\omega t\, d(\omega t) = \frac{100}{\pi}[-\cos\omega t]_0^{\pi}$$

$$= \frac{100}{\pi}[(-\cos\pi) - (-\cos 0)] = \frac{100}{\pi}[(+1) - (-1)] = \frac{200}{\pi}$$

$$= \mathbf{63.66\ volts}$$

Questions 1 The speed v of a vehicle is given by v=4t+3 m/s where t is the time in seconds . Determine the average value of the speed from t=0 to 3 s. (9m/s)

2. Find the mean value of the curve y =6+x + x axis .(4.17)

3. The vertical height h km of a missile varies with the horizontal distance d km is given by h=4d- Determine mean height of the missile from d=0 to d=4km (2.67km)

4. The velocity v of a piston moving with simple harmonic motion at any time t is given by v= c sinωt where is a constant. Determine the mean velocity between t=0 and t= (

(b) Root Mean Square Value: means the squaring a value, then take mean of the value, then the square root

- Root Mean Square Value=
- The principal application of r.m.s value is in alternating currents and voltages.
- It is defined as a value of ac current/voltage that has the same heating effect as a d.c value.

Example 1. Determine the rms value of y=2 x=1 to 4

$$\text{R.m.s. value} = \sqrt{\frac{1}{4-1}\int_1^4 y^2\,dx} = \sqrt{\frac{1}{3}\int_1^4 (2x^2)^2\,dx} = \sqrt{\frac{1}{3}\int_1^4 4x^4\,dx} = \sqrt{\frac{4}{3}\left[\frac{x^5}{5}\right]_1^4} = \sqrt{\frac{4}{15}(1024-1)} = \sqrt{272.8} = \mathbf{16.5}$$

2.A sinusoidal voltage has a maximum value of 100V Calculate its rms value.

Its taken over the range 0 toπ

Q2. Solution

A sinusoidal voltage v having a maximum value of 10 V may be written as: $v = 10\sin\theta$. Over the range $\theta = 0$ to $\theta = \pi$

$$\text{r.m.s. value} = \sqrt{\frac{1}{\pi - 0}\int_0^{\pi} v^2\,d\theta} = \sqrt{\frac{1}{\pi}\int_0^{\pi} (100\sin\theta)^2\,d\theta}$$

$$= \sqrt{\frac{10\,000}{\pi}\int_0^{\pi} \sin^2\theta\,d\theta}$$

Rearranging $\cos 2A = 1 - 2\sin^2 A$ gives $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

$$\text{Hence } \sqrt{\frac{10\,000}{\pi}\int_0^{\pi} \sin^2\theta\,d\theta} = \sqrt{\frac{10\,000}{\pi}\int_0^{\pi} \frac{1}{2}(1 - \cos 2\theta)\,d\theta}$$

$$= \sqrt{\frac{10\,000}{\pi}\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}}$$

$$= \sqrt{\frac{10\,000}{\pi}\frac{1}{2}\left[\left(\pi - \frac{\sin 2\pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right]}$$

$$= \sqrt{\frac{10\,000}{\pi}\frac{1}{2}[\pi]} = \sqrt{\frac{10\,000}{2}} = \frac{100}{\sqrt{2}} = \mathbf{70.71\ volts}$$

The result shows that for

$$\textbf{r.m.s. value} = \frac{1}{\sqrt{2}} \times \textbf{maximum value.}$$

Questions 1. Determine the rms values of (a) y=3x from x=0 to 4 (b) y=m t =1 to 3 © y = 25sinθ from θ= 0 to 2π. (6.928, 4.919,17.68)

2. Calculate the rms values of (a) y=sin 2θ from θ=0 to π/4 (b) y=1+sin t from t=0 to 2π © y=3cos2x from x=0 to π ([0.707,1.225,2.121)

3.The distance p of points from the mean value of a frequency distribution are related to the variable q by the expression p= standard deviation(rms value) correct to 3 s.f for values of from q=1 to 5 (2.58)

4.A current I =30 sin100πt amperes is applied across an electric circuit. Determine its mean and rms values over the range t=0 to 10 ms to 4 s.f. (19.10A, 21.2 A)

5. A sinusoidal voltage has a peak value of 340V.Calculate its mean and rms values correct to 3s.f   (216V, 240V)

6.Dtermine the form factor of a sinusoidal voltage of maximum value f 100 volts given that form factor = (1.11)